UDC 532.59 DOI: 10.26661/2413-6549-2018-2-13

SOME APPLICATIONS OF NUMERICAL INVERSION OF THE LAPLACE TRANSFORM IN PROBLEMS OF PROPAGATION OF WAVE OSCILLATIONS

Selezov Igor, Dr. Prof. Department of Wave Processes

Institute of Hydromechanics NASU,
Sheliabov str., 8/4, Kiev, 03680, Ukraine

igor.selezov@gmail.com

Three algorithms for the numerical inversion of the Laplace transform are considered, for solving specific applied problems of wave dynamics. Comparing the algorithm based on the shifted Lagendre polynomials with the standart solutions shows that there exists an optimal number of terms in expansions. It is also established from the consideration of various algorithms, including modern ones, that the accuracy of all algorithms of numerical inversion decreases with increasing time t (with decreasing in the transformation parameter p in the complex plane). These two conclusions are a consequence of the incorrectness of the inversion Laplace transform problem. The application of the expansion method in the sine arcs to the solution of the initial-boundary value problem (IBV problem) of the study of the propagation of pulse pressure waves in blood vessels is presented. It is based on the equations of the cylindrical shell and blood pressure, and includes the matching conditions at the junction of the vessels, excitation of the pulse pressure wave, its propagation to the junction, reflected and transmitted waves. Application of the same method is presented for the problem of evolution of the free surface of water waves due to local bottom excitation sources that are repeated in time.

Key words: Laplace transform, numerical inversion, IBV problem, algorithm, wave, dynamics, pulse pressure.

ДЕЯКІ ЗАСТОСУВАННЯ ЧИСЕЛЬНОГО ОБЕРНЕННЯ ПЕРЕТВОРЕННЯ ЛАПЛАСА В ЗАДАЧАХ ПОШИРЕННЯ ХВИЛЬОВИХ КОЛИВАНЬ

Селезов Ігор, д. ф.-м. н., професор

Інститут гідромеханіки НАН України,
вул. Желябова, 8/4, Київ, 03680, Україна

igor.selezov@gmail.com

Розглянуто три алгоритми чисельного обернення перетворення Лапласа для розв’язання конкретних прикладних задач динаміки хвиль. Порівняння алгоритму, заснованого на зміщених поліномах Лаґера, зі стандартними розв’язками показує, що існує оптимальне число членів у розкладах. З розгляду різних алгоритмів, у тому числі сучасних, встановлено, що точність всіх алгоритмів чисельної інверсії зменшується зі збільшенням часу t (зі зменшенням параметра перетворення p в комплексній площині). Ці два висновки є наслідком некоректності проблеми перетворення Лапласа. Наведено застосування методу розкладання по дугах синусів до розв'язування початково-крайової задачі (IBV problem) дослідження поширення хвиль пульсового тиску в кровоносних судинах. Вона заснована на рівняннях циліндричної оболонки і кров'яного тиску і включає умови спряження на стику судин, збудження хвилі імпульсного тиску, її поширення до стику, відбиті і прохідні хвилі. Представлено застосування такого ж методу для задачі еволюції вільної поверхні хвиль на воді, обумовленої локальними донними джерелами збудження, які повторюються в часі.

Ключові слова: перетворення Лапласа, чисельне обернення, IBV (початково-крайова задача), алгоритм, хвилі, динаміка, пульсовий тиск.

1. INTRODUCTION

The efficiency of solving problems in mechanics and physics on the basis of the Laplace transform is well known [1–3]. The Laplace transform of a function $f(t)$ is defined by the operator [4, 5]

$$F(p)=\int_0^\infty f(t)e^{-pt}dt \tag{1}$$

for a complex parameter $p = \sigma + i\tau$ under the assumption that $F(p)$ is the analytic function in a domain $\operatorname{Re} p > \sigma_c$, that $F(p)$ converges uniformly in this domain $F(p) \to 0$ relative to $\arg p$ for $p \to \infty$ and that in the case of absolute convergence (1) along the straight line at $\forall \operatorname{Re} p > \sigma_c$ there exists an inversion operator (the Riemann-Mellin integral)

$$f(t) = \frac{1}{2\pi i} \int_{\sigma_c - i\infty}^{\sigma_c + i\infty} F(p) e^{pt} dp. \tag{2}$$

The inverse problem consists in finding the solution $f(t)$ of the integral equation of the first kind (1), where $F(p)$ is a known function of the complex argument p. The kernel e^{-pt} is a smooth function of t and p, the averaging operation f with weight e^{-pt} can substantially smooth out the singularities of the function. The problem of restoring all local irregularities of $f(t)$ requires the involvement of approaches sensitive even to insignificant behavioral features.

The function $f(t)$ is unstable with respect to small variations of $F(p)$. Consequently, the problem of inversion, as the problem of finding the solution $f(t)$ of the integral equation of the first kind (1), belongs to the number of ill-posed problems: solutions are possible not for all quantities of numerical or functional parameters, and weak variations of these parameters can lead to large variations of the solution. This is the main reason that limits the capabilities of all known algorithms of inversion (2).

This paper shows the verification of the expansion method for shifted Legendre polynomials. An application of the method of expansion in even sine arcs to the solution of the IBV problem are given for propagation of pulse pressure waves in blood vessels and water wave generation by local bottom sources of excitation. In addition, some new algorithms are also described.

2. THE METHOD OF EXPANSION IN ORTHOGONAL SHIFTED LEGENDRE POLYNOMIALS [6]

The introduction of the transformation $e^{-t} = \zeta$ takes the interval $(0, \infty)$ of a variable t into an interval $(0,1)$ of a variable ζ. After this expression (1) takes the form

$$F(p) = \int_0^1 f(\zeta) \zeta^{p-1} d\zeta, \tag{3}$$

and the function $f(\zeta)$ (2) is represented in the form of a convergent series in polynomials orthogonal on a segment $[0,1]$ which are given by the shifted Legendre polynomials $P_n^*(\zeta)$

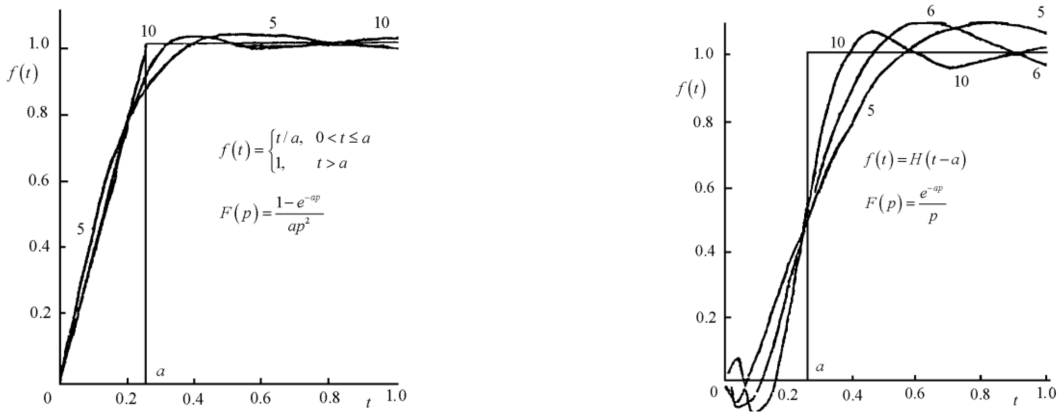
$$f(\zeta) = \sum_{n=0}^{\infty} (2n+1) a_n P_n^*(\zeta), \quad a_n = \sum_{k=0}^{n} \alpha_k^{(n)} F(k+1), \tag{4}$$

where

$$P_n^*(\zeta) = (-1)^n \sum_{k=0}^{n} \alpha_k^{(n)} \zeta^k, \quad \alpha_k^{(n)} = (-1)^k \binom{n}{k} \frac{(n+k)!}{n!k!}.$$

The function $F(k+1)$ in (4) with integer argument corresponds to the function $F(p)$ of (1).

The coefficients a_n on the basis of (3) and (4) are calculated in a finite number of equidistant points k along the real axis of the transformation parameter p (Fig. 1). Calculations were carried out for the number of terms of the series 5, 6, ..., 10, and the number 10 was found to be optimal for approximation with the number of significant digits equal to 9. In all cases, the accuracy increases at n up to 10.



a) b)

Fig. 1. The results of numerical inversion by the shifted Legendre polynomials:
a) for a function linearly increasing, and then constant; b) for the Heaviside function

Similar results were obtained for table functions linearly or instantaneously increasing, and then decaying exponentially.

On the basis of this algorithm, a number of IBV problems were solved in the theory of wave propagation and diffraction including for a water hammer, the effect of a pulse on an elastic shell, the generation of tsunami waves and others.

3. THE METHOD OF EXPANSION IN THE SINE ARCS

This method was proposed in [7] and is described in [2]. In this case, the solution is represented in the form of a series

$$\Phi(\theta)=\sum_{\nu=0}^{\infty} C_\nu \sin(2\nu+1)\theta \tag{5}$$

under the condition that $e^{-\sigma t}=\cos\theta$, $f(t)=f\left(-\frac{1}{\sigma}\ln\cos\theta\right)=\Phi(\theta)$, $p=(2n+1)\sigma$, $\sigma>0$, $n=0,1,\ldots$. As a result, we obtain the linear system of equations to determine the coefficients

$$C_\nu : C_0=\frac{4}{\pi}\sigma f^L(\sigma),\ C_0+C_1=\frac{4^2}{\pi}\sigma f^L(3\sigma),\ 2C_0+3C_1+C_2=\frac{4^3}{\pi}\sigma f^L(5\sigma),\ \ldots .$$

It should be noted that the implementation of this algorithm essentially depends on the value of the parameter σ in (5), the optimal choice of which requires conducting numerical experiments.

Let us consider the problem of propagation of pulse pressure waves in blood vessels [8]. In this case, a function that approximates the cardiac pulse is given in the form $f(t)=te^{-\alpha t}$.

The corresponding hydroelasticity problem is formulated as the IBV problem for differential equations in three regions, including the matching conditions in two sections $x=-x_1$ and $x=0$, the conditions at infinity for $x\to-\infty$ and $x\to\infty$ and also the initial conditions for $t=0$.

We consider two semi-infinite elastic cylindrical shells of constant, but different thickness, filled with a inviscid incompressible fluid, ideally conjugated in a section $x=0$. The left shell occupies the region $\Omega_h^f(-\infty<x<0)$, the right shell – the region $\Omega_d(0\le x<\infty)$. The left region $\Omega_h(0\le x<\infty)$ is divided into two subregions $\Omega_h'(-\infty<x<x_1)$ and $\Omega_h(-x_1\le x<0)$ by a cross section $x=-x_1$, in which a pressure pulse $f(t)$ propagating to the left and to the right at a time

$t=0$ is specified. The pulse, reaching the junction of the shells $x=0$, generates the reflected wave to the left and the passing wave through the junction to the right. The stress concentration in the interface junction is of primary interest in the passage of the pulse, and here it is investigated on the basis of the Laplace transform and the algorithm for numerical inversion. It is assumed that the motion of the shells is described by the theory of Kirchhoff shells, and the motion of the liquid inside the shells by a quasi-one-dimensional model.

The heart pressure pulse with a doubled amplitude is applied on the left on some distance from the junction of the vessels. One half of this pulse spreads to the left. Another part that also very accurately approximates the heart pulse $f(t)$ runs to the right. After reaching the junction of the vessels this pulse partially reflected from the junction and partially transmitted through to the right.

For each of the three regions, a system of differential equations for radial displacement w and for pressure p as functions of time t and axial coordinate x is written.

The motion of the shell is described by a differential equation [9]

$$D\frac{\partial^4 w}{\partial x^4}+\frac{Eh}{a^2}w+\rho h\frac{\partial^2 w}{\partial t^2}=\hat{p}\,, \tag{6}$$

and fluid motion by the system of equations [10]

$$\frac{\partial^2 w}{\partial t^2}=\frac{r_i}{2}\frac{1}{\rho_f}\frac{\partial^2 \hat{p}}{\partial x^2}, \quad p_i=\hat{p}-\frac{r_i^2}{8}\frac{\partial^2 \hat{p}}{\partial x^2}, \quad \frac{\partial u}{\partial t}=-\frac{1}{\rho_f}\frac{\partial \hat{p}}{\partial x}. \tag{7}$$

The matching conditions are of the form:

on the interface $x=-x_1$ (impulse application)

$$\hat{p}_h'-\hat{p}_h=f(t)\,, \quad u_h'-u_h=0, \quad w_h'-w_h=0, \tag{8}$$

$$\frac{\partial w_h'}{\partial x}-\frac{\partial w_h}{\partial x}=0, \quad \frac{\partial^2 w_h'}{\partial x^2}-\frac{\partial^2 w_h}{\partial x^2}=0, \quad \frac{\partial^3 w_h'}{\partial x^3}-\frac{\partial^3 w_h}{\partial x^3}=0\,; \tag{9}$$

at the interface $x=0$ (vessel junction)

$$\hat{p}_h-\hat{p}_d=0, \quad u_h-u_d=0, \quad w_h-w_d=a_d-a_h, \tag{10}$$

$$\frac{\partial w_h}{\partial x}-\frac{\partial w_d}{\partial x}=0, \quad D_h\frac{\partial^2 w_h}{\partial x^2}-D_d\frac{\partial^2 w_d}{\partial x^2}=0, \quad D_h\frac{\partial^3 w_h}{\partial x^3}-D_d\frac{\partial^3 w_d}{\partial x^3}=0\,. \tag{11}$$

The results calculations for real vessel parameters are presented on Fig.2

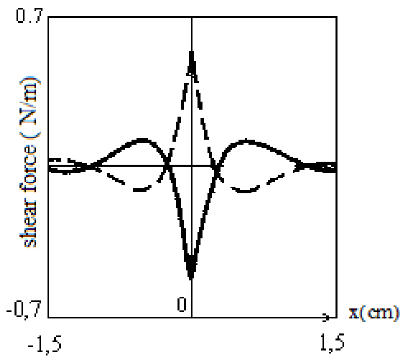


Fig. 2. Concentration of shear force at $x=0$

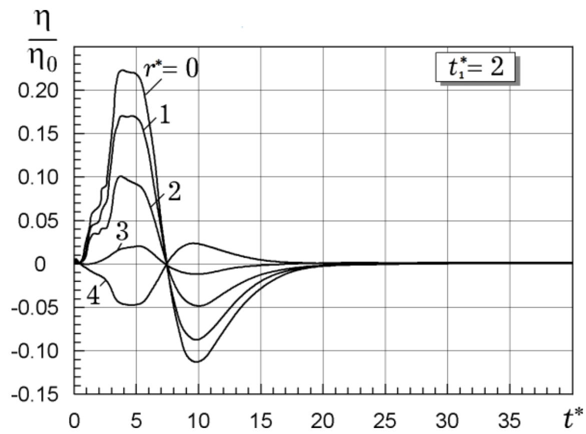


Fig. 3. Free water surface at different distances $r^* = r/r_0$ from epicenter $\eta = 0$ in the case of repeated excitation time $t_1^* = 2$ (r_0 is the radius of convex axially symmetric bottom lift, $t^* = t\sqrt{gH_0}/r_0$, H_0 is the water height)

The method was used in investigations of a free surface evolution at local bottom repeated in time excitations for analysis of tsunami waves [11]. The results of calculations are presented in Fig. 3.

4. OTHER APPROACHES AND SOME RECENT ONES

We note the simplest approximation - the expansion in exponential functions [12], the Fourier-Bessel expansion method [13], based on the representation of the solution in the form of Fourier-Bessel expansions and the regularization of an ill-posed problem [14]. The coefficients of such an expansion are presented in the form of well-convergent series, which makes it possible to calculate the values of the smooth original function with high accuracy.

The function $f(t)$ is represented as a series $f_A(t) = \sum_{i=1}^{N} A_i e^{-t/t_i}$ where the values A_i are undetermined coefficients, and the quantities t_i are given positive constants, so that $f_A(t)$ is an approximate representation of the function $f(t)$. The total quadratic error, determined by the difference between $f(t)$ and $f_A(t)$ is written in the form $E^2 = \int_0^\infty \left[f(t) - f_A(t) \right]^2 dt$. The coefficients A_i are determined from the condition of the minimum of the total root-mean-square error $\int_0^\infty \left[f(t) - f_A(t) \right]^2 e^{-t/t_i} dt = 0$. These relations form a system N of linear algebraic equations for the determination of N of unknown coefficients A_j. It is established from the calculations that acceptable accuracy can be achieved in a small interval of variation t.

Numerous papers have been devoted to the investigation of the numerical inversion of the Laplace transform. In [6] an application of the Chebyshev, Laguer and Jacobi polynomials was considered, which were subsequently applied in [15, 16]. We note some asymptotic methods, the numerical inversion with the use of Laguer polynomials was considered in [16], Jacobi polynomials in [15], in problems of mechanics and physics in [16-18].

A characteristic feature of all methods of numerical inversion is that they are well realized for large parameters of the Laplace transform $|p| < \infty$, $|p| >> 1$, that is, near an infinitely distant point in the complex plane $p = \operatorname{Re} p + i \operatorname{Im} p$. As the value $|p|$ decreases, all methods of numerical conversion deteriorate.

An alternative approach to the numerical inversion of the Laplace transform based on Fourier expansions [19] was used to solve the IBV problem of thermoelasticity for a hyperbolic system of equations involving a higher-order hyperbolic operator (fourth) than the hyperbolic heat transfer operator (the second) taking into account relaxation time [20].

We note the method of numerical inversion presented in [21]. This method has been used in [22] to solve a new IBV problem on a finite interval for a hyperbolic equation with relaxation parameter (parabolic operator). The original is determined by the formula

$$T(x,t) = \frac{10^{M/3}}{t}\sum_{k=0}^{2M}\eta_k \operatorname{Re}\left(T^L\left(x, \frac{\beta_k}{t}\right)\right),$$

where $\beta_k = \frac{M\ln(10)}{3} + \pi i k$, $\eta_k = (-1)^k \xi_k$, $\xi_0 = \frac{1}{2}$, $\xi_k = 1$, $1 \le k \le M$, $\xi_{2M} = \frac{1}{2^M}$, $\xi_{2M-k} = \xi_{2M-k+1} + 2^{-M} C_M^k$, $0 < k < M$.

The value of the parameter M in (15) was assumed to be 16, which, according to [23], gives accuracy in nine significant digits. The superscript L notes the Laplace transform.

5. CONCLUSIONS

Some methods of numerical inversion of the Laplace transform and their applications to the solution of IBV problems have been considered. Algorithms, some comparisons with exact solutions, estimates of accuracy are given. The results of the study of the propagation of pressure pulse waves in blood vessels and the generation of waves on water by underwater earthquakes are presented. Some recent approaches have been noted.

Acknowledgments

The author thanks S. A. Savchenko for preparing the manuscript and article.

REFERENCES

1. Carslaw H. S., Jaeger J. C. Operational methods in applied mathematics. Oxford: The Clarendon Press, 1941.
2. Doetsch T. Anleitung zum praktischen Gebrauch der Laplace-Transformation. Muenchen: R. Oldenbourg, 1956.
3. Lurie A. I. Operational calculus and its applications to problems of mechanics (in Russian). Moscow; Leningrad: Gos. Izd. Fiz.-Mat. Lit., 1960. 208 p.
4. Gnoenski L. S., Kamenski L. E., Elsgolz L. E. Mathematical foundation of the theory of controlled systems (in Russian). Moscow: Nauka, 1969. 512 p.
5. Krylov V. I., Skoblya N. S. Methods of approximate Fourier transform and inverting Laplace transform. Moscow: Nauka, 1974. 224 p.
6. Lanczos C. Applied analysis. New Jersey: Prentice Hall, 1956.
7. Papoulis A. A new method of inversion of the Laplace transform. *Quart. Appl. Math.* 1957. N 14. P. 405-414.
8. Selezov I., Fratamico G. Pulse waves in arteries with a vessel joint. *Ed. M. Mela and G. Pallotti, Int. Summer School in Biophysics*. (Italy, Bellaria Igea Marina, 9–16 September 1996). Italy, Bellaria Igea Marina, 1996. P. 89–94.
9. Timoshenko S., Woinowsky-Krieger S. Theory of plates and shells. *McGraw-Hill Book Company, INC* 1959.
10. Moodie E. B., Barday D. W., Tait R. T. A boundary value problem for fluid-filled viscoelastic tubes. *Mathematical Model*. 1983. N 4. P. 195–207.
11. Selezov I. T., Kuznetsov V. V., Chernikov D. O. Generation of surface gravity waves by bottom time-repetitive pulses. *J. Math. Sci.* 2010. **171**, N 5. P. 596–602.

12. Schapery R. A. Approximate methods of transform inversion for viscoelastic stress analysis. *Proc. 4th U.S. Nat. Congr. Appl. Mech*. 1962. P. 1075–1085.

13. Selezov I. T., Korsunski S. V. Numerical inverting Laplace transform based on Fourier-Bessel expansions (in Russian). *Dokl. Acad. Nauk UkrSSR, Ser. A*. 1988. N 11. P. 25–28.

14. Tikhonov A. N., Arsenin V. J. Methods of solution of ill-posed problems (in Rissian). Moscow: Nauka, 1986. 288 p.

15. Miller M. K., Guy W. T. Numerical inversion of the Laplace transform by use of Jacobi polynomials. *SIAM J. Numer. Analysis*. 1966. **3**. P. 624–635.

16. Weeks W. T. Numerical inversion of Laplace transforms using Laguerre functions. *J. Assoc. Comput. Math*. 1966. **13**. P. 419–426.

17. Vilge J. I., Zakharov E. V. To calculation of nonstationary acoustic field in slit by the method of numerical inversing Laplace transform (in Russian). *Collection chisl. Metody in geophys*. 1978. N 1. P. 102–107.

18. Bellman R., Lockett R. E., Kalaba J. A. Numerical inversion of the Laplace transform: applications to biology, economics and physics. Amsterdam, Elsevier 1966.

19. Honig G., Hirdes U. A method for the numerical inversion of Laplace transform. *J. Comp. App. Math*. 1984. **10**, N 1. P. 113–132.

20. Sherief H. H., Youssef H. M. Short time solution for a problem in magnetothermoelasticity with thermal relaxation. *J. Thermal stresses*. 2004. **27** (6). P. 537–559.

21. Abate J., Whitt, W. A unified framework for numerically inverting Laplace transforms. *INFORMS J. on Computing*. 2006. **18**, N 4. P. 408–421.

22. Selezov I. T., Kryvonos Ju. G. Modeling medicine propagation in tissue: generalized statement. *Cybernetics and Systems Analysis*. 2017. **53**, № 4. P. 535–542. DOI: 10.1007/s10559-017-9955-1.

УДК 539.3 DOI: 10.26661/2413-6549-2018-2-14

ПОЗДОВЖНІЙ МОДУЛЬ ПРУЖНОСТІ ВОЛОКНИСТОГО КОМПОЗИТА З ПЕРЕХІДНИМ ШАРОМ

Столярова А. В., аспірант, Гребенюк С. М., д. т. н., доцент, Клименко М. І., к. ф.-м. н., доцент

Запорізький національний університет,
вул. Жуковського, 66, м. Запоріжжя, 69600, Україна

st_av991@ukr.net, gsm1212@ukr.net, m1655291@gmail.com

При розв’язанні задач механіки композитів зручно використовувати модель композита у вигляді суцільного однорідного середовища з ефективними сталими, що адекватно відображають його найбільш суттєві характеристики. Задачі визначення напружено-деформованого стану композита базуються на припущенні, що з’єднання матриці з волокном має чітку границю розподілу, що обмежує компоненти композита. Однак важливу роль у механіці композитів відіграє ефект неідеального контакту між компонентами, одним із яких є наявність перехідного шару. Отримано формулу залежності поздовжнього модуля пружності для транстропного матеріалу, що моделює композит, від пружних характеристик матриці, волокна, перехідного шару, що утворюється між матрицею та волокном, та об’ємної долі кожного з них у композиті. Для цього розв’язано дві крайові задачі: про поздовжнє розтягування нескінченного складеного ізотропного тришарового циліндра та поздовжнє розтягування нескінченного транстропного суцільного циліндра. Для розв’язування системи рівнянь рівноваги в переміщеннях у циліндричній системі координат у роботі використані наступні припущення: матеріали матриці, перехідного шару і волокна є ізотропними, площини ізотропії співпадають та перпендикулярні осі волокна; задача