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MODEL OF HOMOGENIZATION OF THE MULTI-MODULAR TRANSTROPIC FIBROUS COMPOSITE

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In this article we solve the problem of following effective mechanical constants determination: the transverse elasticity modulus and the Poisson's ratio in the plane of isotropy for the transtropic composite. We consider the fibrous unidirectional composite material composed of the isotropic elastic matrix and the fiber. We assume that mechanical characteristics of components are different for stretching and compression. So materials of the matrix and the fiber are multi-modular. For composite's properties modeling we consider its unit cell. It is an endless cylinder. It is made of a solid cylinder as a fiber, and a hollow cylinder as a matrix. On the contacting surface between the matrix and the fiber the relative torsion angle is continuous. Composite's material is modeled by a solid homogeneous transversally-isotropic multi-modular material. Its isotropy plane is perpendicular to the fiber axis. We solve the problem of torsion of the cylinder unit cell under the action of constant torque applied to determine the effective transverse shift module of composite. The non-zero component of the stress-strain state of composite's cell is a tangent stress in the plane of isotropy. We find the relative torsion angle for the matrix and the fiber. We also solve the similar problem for the homogeneous transversally-isotropic cylinder cell as a composite. We determine the shift module according to kinematic terms of the displacement of a relative torsion angle on an outer matrix surface and the same angle of th eouter surface of a cell of a homogeneous composite. The transverse shift module of the composite we use to determine other effective constants: the transverse elasticity modulus and the Poisson's ratio in the plane of isotropy of the composite. These constants are functions of mechanical characteristics of the matrix and the fiber, also depend on the volume fraction of the fiber in the composite. Obtained formulas could be used for the stress-strain state calculation of composite constructions.

МОДЕЛЬ ГОМОГЕНІЗАЦІЇ РІЗНОМОДУЛЬНОГО ТРАНСТРОПНОГО ВОЛОКНИСТОГО КОМПОЗИТУ

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Ключові слова: *різномодульна теорія пружності, композит, модуль зсуву, матриця, волокно, кручення, відносний кут закручування, умови узгодження.*

У роботі виконано задання з визначення таких ефективних механічних сталих, як поперечний модуль пружності та коефіцієнт Пуассона у площині ізотропії транстропного композиту. Розглянуто волокнистий односпрямований композит, що складається з ізотропних пружних матриці та волокна. Припускається, що під час розтягування та стискання механічні властивості компонентів відрізняються між собою, тобто матеріали матриці та волокна є різномодульними. Для моделювання властивостей композитного матеріалу використовується його елементарна комірка. Вона є нескінченним циліндром. Він складається із суцільного циліндра, що моделює волокно, вкладеного в порожнинний циліндр, що моделює матрицю. На межі контакту матриці та волокна відносний кут закручування вважаємо неперервним. Матеріал композиту моделюється суцільним однорідним трансверсально-ізотропним різномодульним матеріалом. Його площина ізотропії перпендикулярна осі волокна.

Для визначення ефективного поперечного модуля зсуву композиту розв'язується завдання кручення циліндричної елементарної комірки під дією прикладеного до неї сталого крутного моменту. Ненульовим складником напружено-деформованого стану комірки композиту є дотичне напруження, що діє у площині ізотропії. Визначається відносний кут закручування для матриці та волокна. Аналогічне завдання розв'язане для однорідної трансверсально-ізотропної циліндричної комірки, що моделює композит. Модуль зсуву визначається з кінематичної умови узгодження відносного кута закручування на зовнішній поверхні матриці та значення цього кута на зовнішній поверхні представницької комірки однорідного композиту. Знайдений поперечний модуль зсуву було застосовано для визначення таких ефективних сталих, як поперечний модуль пружності та коефіцієнт Пуассона у площині ізотропії композиту. Ці співвідношення отримано у вигляді функцій механічних характеристик матриці та волокна, а також об'ємної частки волокна в матеріалі композиту. Визначені у роботі ефективні пружні сталі можна використовувати для розрахунку напружено-деформованого стану елементів конструкцій, виготовлених із композитів.

Introduction. Mechanical characteristics of composite materials are necessary to determine the stress-strain state. Therefore, problem of homogenization of a composite implies calculation of its mechanical characteristics as homogeneous materials. In many cases, this problem also complicated by differences in meanings of these characteristics depends on character of material's deformation. Particularly elastic constants of composite are different under tension and compression. In these cases of homogenization of a composite, we use models of the multi-modular theory of elasticity.

Different models of the multi-modular environment are developed in researches [1; 2]. Results of experimental researches of multi-modular grainy composites are summarized in [3; 4]. Author of [5] considers isotropic multi-modular composites. The phenomenon of multi-modularity for reinforced concrete is developing in [6].

Solution of the problem of homogenization of transversally-isotropic elastic composites regardless multi-modularity is in monograph [7]. Defining of effective elastic components of multi-modular transtropic composite includes in [8; 9]. Meanwhile, in [8] considered the formula for effective elastic constant as transverse module of elasticity Type I and Poisson's ratio in the plane of isotropy of composite. In [9] we determined longitudinal module of elasticity under tension and compression of multi-modular composite.

We analyzed publications and come to the conclusion that problem of determine of complete system of elastic constants for its mechanical characteristics are unsolved yet. The main purpose of following research is to determine effective characteristics as the transverse modulus of elasticity and the Poisson's ratio in the plane of isotropy of multi-modular transtropic composite. We use methods proposed in [7]. In this case we have to solve problem of transverse shift for system "matrix – fiber" and for similar homogeneous composite's cell. However, at the same time appears difficulties depend on multi-modularity of materials. In this article, we solve problems considering problem of torsion of elementary cylinder composite's cell. So we can find the effective transverse shift module of the composite. According to it, we determine required constants.

Problem statement. We consider the problem of homogenization of transtropic fibrous composite with multi-modular properties. It includes following isotropic elastic multi-modular components: matrix and fiber. In [8] we found the ratio $\frac{1-\nu_{23}}{E_2}$ for tension and compression. E_2 is the elasticity module and ν_{23} is the Poisson's ratio for the isotropy plane of composite. To obtain formulas for its constants we define an effective shift module as following formula:

$$G_{23} = \frac{E_2}{2(1+\nu_{23})}. \tag{1}$$

Elementary composite's cells are made of a solid cylinder $(0 \le r \le a)$, as a fiber, and a hollow cylinder $(a \le r \le b)$, as a matrix. We model the composite as a cylinder $(a \le r \le b)$ made of a homogeneous transtropic material. The isotropy plane is perpendicular to the cylinder cell's axial. We solve two tasks to determine the effective shift module G_{23}. Firstly, we solve the problem of torsion of a compound cylinder making the system "matrix – fiber". Secondly, we solve the problem of torsion of a homogeneous cylinder modeling a composite.

Determination of the effective shift module in the isotropy plane of the transtropic composite. We suppose that the twisting moment M is attached to the outer surface of elementary composite's cell. We consider the cross section of the cell perpendicular to its axial. The torque cased field of the tangent stress at each point. In this case, the tangent stress is divided linearly over the cylinder thickness. Symbol * is used for a matrix and symbol * is for a fiber. We use the cylindrical coordinate system (r, θ, z), where z-axis the same to the cylinder axial. The tangent stress at each points of a fiber can be written as:

$$\tau^{*} = \tau_a \cdot \frac{r}{a}, \tag{2}$$

In our notation r represents radial coordinates (the distance between point to the cylinder axis), a is the fiber radius, τ_a is the tangent stress for $r = a$.

The relative torsion angle $\frac{d\varphi}{dz}$ depends on the tangent stress as the following formula:

$$\frac{d\varphi}{dz} = \frac{\tau}{G \cdot r}. \tag{3}$$

In (3) $G = \dfrac{E}{2(1+\nu)}$ is the shift module.

For the relative torsion angle at the point of matrix and fiber equality holds:

$$\frac{d\varphi^{\circ}}{dz} = \frac{\tau^{\circ}}{G^{\circ} \cdot r}, \quad \frac{d\varphi^{*}}{dz} = \frac{\tau^{*}}{G^{*} \cdot r}. \tag{4}$$

On the surface of the contact between the matrix and the fiber $r = a$ the relative torsion angle is a continuous function, so we have

$$\left.\frac{d\varphi^{\circ}}{dz}\right|_{r=a} = \left.\frac{d\varphi^{*}}{dz}\right|_{r=a}. \tag{5}$$

From the eq. (4) and (5) using the equality $r = a$ in tangent tensions for matrix and fiber, the following formula could be obtained:

$$\tau^{*}(a) = \frac{G^{*}}{G^{\circ}} \tau^{\circ}(a) = \frac{G^{*}}{G^{\circ}} \tau_a. \tag{6}$$

In the matrix, the tangent stress is also disturbed by the linear principle: $\tau^* = k \cdot r$. We determine the constant k from the following condition:

$$\tau^*(a) = ka = \frac{G^*}{G^\circ} \cdot \tau_a \qquad (7)$$

The formula for the coefficient k can be written as:

$$k = \frac{G^* \tau_a}{G^\circ \cdot a}. \qquad (8)$$

We obtain the tangent stress at points of matrix as follows:

$$\tau^* = \frac{G^*}{G^\circ} \cdot \frac{r}{a} \cdot \tau_a. \qquad (9)$$

The twisting moment M, put in the outer surface of cylinder cell $r = b$, add the tangent stress τ_b. So we obtain the following formula:

$$\tau^* = \tau_b \cdot \frac{r}{b}. \qquad (10)$$

We denominate the tangent stress τ_b using the tension τ_a:

$$\tau^*(a) = \tau_b \cdot \frac{a}{b} = \frac{G^*}{G^\circ} \tau_a \Rightarrow \tau_b = \frac{G^*}{G^\circ} \cdot \frac{b}{a} \cdot \tau_a. \qquad (11)$$

The formula (10) for the tangent stress in the matrix can be written as:

$$\tau^* = \frac{G^*}{G^\circ} \cdot \frac{r}{a} \cdot \tau_a. \qquad (12)$$

We define a formula for the twisting moment M using the equality $r = b$ as the following formula:

$$M = 2\pi \int_0^a \tau^\circ r^2 dr + 2\pi \int_a^b \tau^* r^2 dr = 2\pi \int_0^a \tau_a \cdot \frac{r^3}{a} dr + \\ +2\pi \int_a^b \tau_a \cdot \frac{G^*}{G^\circ} \cdot \frac{r^3}{a} dr = \frac{\pi \cdot \tau_a}{2a} \left[a^4 + \frac{G^*}{G^\circ} \left(b^4 - a^4 \right) \right]. \qquad (13)$$

Consider the torsion of the homogeneous transtropic cylinder adding the twisting moment M and the equality $r = b$ to it. Hence the relative torsion angle could be obtained as:

$$\frac{d\varphi}{dz} = \frac{M}{G_{23} \cdot J_r} = \frac{2M}{\pi \cdot G_{23} \cdot b^4}. \qquad (14)$$

If $r = b$ then the relative torsion angle of the twisting matrix is equal to the relative torsion angle of twisting on outer surface of the homogeneous cylinder.

Using the formula (13), we obtain the previous angle:

$$\left. \frac{d\varphi}{dz} \right|_{r=b} = \frac{2M}{\pi \cdot G_{23} \cdot b^4} = \frac{\tau_a}{a \cdot G_{23}} \left[f^2 + \frac{G^*}{G^\circ} \left(1 - f^2 \right) \right]. \qquad (15)$$

In (15) $f = \frac{a^2}{b^2}$ is the volume ratio of the fiber in the composite. The relative torsion angle of the twisting matrix for $r = b$ could be written as follows:

$$\left. \frac{d\varphi^*}{dz} \right|_{r=b} = \frac{\tau^*}{G^* b} = \frac{\tau_a}{a \cdot G^\circ}. \qquad (16)$$

We equate (15) and (16). So we find the effective transverse shift module as

$$G_{23} = G^\circ f^2 + G^* \left(1 - f^2 \right). \qquad (17)$$

So we define the ratio:

$$\frac{E_2}{2\left(1 + \nu_{23}\right)} = G^\circ f^2 + G^* \left(1 - f^2 \right), \qquad (18)$$

E_2 and ν_{23} are effective values of the transverse elasticity modulus and the Poisson's ratio in the plane of isotropy of the composite.

Defining the effective shift module and the poisson's ratio in the plane of isotropy of the composite. All components of tensions among the tangent stress in the plane of isotropy are equal to the zero while torsion. Therefore, G_{23} is value under tension and compression have to be equal:

$$\frac{E_2^+}{2\left(1 + \nu_{23}^+\right)} = \frac{E_2^-}{2\left(1 + \nu_{23}^-\right)} = \frac{E_2}{2\left(1 + \nu_{23}\right)}. \qquad (19)$$

In our notation the symbol "+" is used for tension and the symbol "–" is used for compression. In (8) we determined ratio between the transverse shift module and the Poisson's ratio under tension and compression. In the case of tension, we have the following equation:

$$\frac{1 - \nu_{23}^+}{E_2^+} = \frac{\left(1 - \nu_+^o\right)\left(f\left(1 + \nu_+^*\right) + \left(1 - \nu_+^*\right)\right)}{d_2 - d_1} + \frac{E_+^o\left(1 + \nu_+^*\right)\left(1 - \nu_+^*\right)(1 - f)}{E_+^*\left(d_2 - d_1\right)} + \\ + \frac{d^o f(f - 1)\left(\nu_+^* E_+^o\left(1 + \nu_+^*\right) + E_+^*\left(\nu_+^*\left(1 - \nu_+^o\right) - 2\nu_+^o\right)\right)}{E_+^*\left(d_2 - d_1\right)\left(d^* f + d^o(1 - f)\right)} \qquad (20)$$

Denote

$$d_0 = \frac{4\nu_+^\circ E_+^* - 2\nu_+^*\left(E_+^\circ\left(1 + \nu_+^*\right) + E_+^*\left(1 - \nu_+^\circ\right)\right)}{E_+^*},$$

$$d_1^\circ = -d^\circ = \frac{\left(E_+^\circ\left(f\left(1 - \nu_+^* - 2\nu_-^*\nu_+^*\right) + \left(1 + \nu_+^*\right)\right) + E_+^*(1 - f)\left(1 - \nu_+^\circ - 2\nu_+^\circ\nu_-^\circ\right)\right)}{E_-^\circ}$$

$$d_1^* = d^* = \frac{\left(E_+^*(f - 1)\left(1 - \nu_+^\circ - 2\nu_-^*\nu_+^*\right) - E_+^\circ\left(f\left(1 - \nu_+^* - 2\nu_+^*\nu_-^*\right) + \left(1 + \nu_+^*\right)\right)\right)}{E_-^*}$$

The right part of (18) under tension we denote as A_1^+, and the right part of (20) we denote as A_2^+:

$$A_1^+ = \frac{E_+^\circ}{2\left(1 + \nu_+^\circ\right)} \cdot f^2 + \frac{E_+^*}{2\left(1 + \nu_+^*\right)}\left(1 - f^2\right). \qquad (21)$$

Thus we have the following system of equations:

$$\begin{aligned} \frac{E_2^+}{2\left(1 + \nu_{23}^+\right)} &= A_1^+, \\ \frac{1 - \nu_{23}^+}{E_2^+} &= A_2^+. \end{aligned} \qquad (22)$$

Finally, we get from this system:

$$E_2^+ = \frac{4A_1^+}{1 + A_1^+ A_2^+}, \tag{23}$$

$$\nu_{23}^+ = \frac{1 - 2A_1^+ A_2^+}{1 + 2A_1^+ A_2^+}. \tag{24}$$

To obtain values of other effective constants under compression we have to change in formulas (20) – (23) the index symbol to the opposite sign.

Thus, we determined formulas for effective constants of the transtropic fibrous composite such as the transverse elasticity modulus and the Poisson's ratio in the plane of isotropy of the composite.

Conclusions. Thus, we obtained formulas that gave us a possibility to define effective characteristics of the multi-modular composites such as the transverse elasticity modulus and the Poisson's ratio in the plane of isotropy of the composite. So we devised formulas that complemented the system of analytic formulas for the effective characteristics of the multi-modular composites in [8; 9]. To obtain a complete system for all elastic effective constants properly-related parameters as multi-modular transtropic composites, we have to define analytic formulas for the transverse shift G_{12}. We hope to solve this problem in future articles.

Note, obtained formulas for the effective values E_2 and ν_{23} without multi-modularity of the matrix and the fiber are the same to formulas for transtropic composites in [7].

We can conclude that to define the stress-strain state of constructions element made of composites we have to know effective constants of their characteristics. While homogenization of composites we have to consider differences in their mechanical characteristics under tension and compression available for some materials.

For solving the problem of determination of effective constants of fibrous composite, we can take into terms conditions of displacements of homogeneous composite and its components. We may define the transverse elasticity modulus and the Poisson's ratio in the plane of isotropy of the composite according to the conditions of the equality of the relative torsion angle on the outer surface of a matrix and the elementary homogeneous composite cell.

The opportunity for future researches in the field of homogenization of the transtropic multi-modular composites relates with formulas for G_{12} and development of improved mathematical models for defining effective mechanical characteristics of the materials.

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