РОЗДІЛ І. ПРИКЛАДНА МАТЕМАТИКА

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ESTIMATION OF MAXIMUM LIKELIHOOD IN THE CONDITIONS OF INSPECTIONS OF THE CURRENT STATE

Makarov V. A.
Doctor of Engineering Sciences, Docent
Transport and Telecommunication Institute
Lomonosov str., 1, LV-1019, Riga, Latvia
makarovs.valerijs@gmail.com

Kazakova A. V.
Riga State Gymnasium No. 2
Krisjana Valdemara str., 1, LV-1010, Riga, Latvia
kazakova.anastasija@gmail.com

Key words: *observations of current state, initial damage, fully censored samples, maximum likelihood method, distribution parameters, hypothesis testing.*

The article discusses the algorithm and methods of statistical processing of one type of grouped data – interval-censored samples with overlapping observation intervals. The samples are censored by the observation time interval of the measured value. The samples consist entirely of such interval observations.
The initial information was the data of one-time and multiple inspections of the structure in operation in order to detect damage in the form of cracks. The stage of appearance of the initial crack (macrocrack) in the zone of stress concentration is determined. A macrocrack, for objective reasons, cannot be directly detected during inspections. Examples of describing the time of formation of initial cracks in the form of interval-censored samples with intersecting observation intervals are presented.
The main focus was on the development of methods for statistical processing of the studied samples. The maximum likelihood method was applied for parametric estimation of the time of appearance of macrocracks. Basic maximum likelihood equations are obtained for distributions with location and scale parameters. The computational procedure of the accumulation method with the use of Fisher's information is applied. Expressions for calculating the variances of parameter estimates are obtained. An equation for checking the existence of a solution to the system of likelihood equations is presented. The appearance of macrocracks can have a different physical nature, for example, the accumulation of fatigue or instantaneous damage due to an overload exceeding the calculated one. Different physical nature implies different types of preventive work to prevent them and is described by different probabilistic models. A likelihood ratio criterion is proposed to test the statistical hypothesis about the situation of sudden appearance of macrocracks at a load exceeding the calculated one.
The results obtained show the features of the structure of the studied samples and possible methods of statistical processing. Bibliography 11 titles Table 2.

ОЦІНКА МАКСИМАЛЬНОЇ ІМОВІРНОСТІ В УМОВАХ ПЕРЕВІРОК СУЧАСНОГО СТАНУ

Макаров В. А.
доктор технічних наук, доцент
Інститут транспорту та зв'язку
Ломоносова, 1, LV-1019, Рига, Латвія
makarovs.valerijs@gmail.com

Казакова А. В.
Ризька державна гімназія № 2
Криш'яна Валдемара, 1, LV-1010, Рига, Латвія
kazakova.anastasija@gmail.com

Ключові слова:
спостереження поточного стану, початкові пошкодження, повністю цензуровані вибірки, метод максимальної правдоподібності, параметри розподілу, перевірка гіпотези.

У статті розглянуто алгоритм і методи статистичної обробки одного типу згрупованих даних – інтервально-цензурованих вибірок із перекриттям інтервалів спостереження. Зразки цензуруються за інтервалом часу спостереження вимірюваного значення. Вибірки повністю складаються з таких інтервальних спостережень.

Вихідною інформацією були дані одноразових та багаторазових обстежень конструкції, що експлуатується, з метою виявлення пошкоджень у вигляді тріщин. Визначено стадію появи початкової тріщини (макротріщини) в зоні концентрації напружень. Макротріщину з об'єктивних причин не можна безпосередньо виявити під час оглядів. Наведено приклади опису часу утворення початкових тріщин у вигляді інтервально-цензурованих вибірок з пересічними інтервалами спостереження.

Основну увагу було приділено розробці методів статистичної обробки досліджуваних вибірок. Для параметричної оцінки часу появи макротріщин застосовано метод максимальної правдоподібності. Отримано основні рівняння максимальної правдоподібності для розподілів з параметрами розташування та масштабу. Застосовується обчислювальна процедура методом накопичення з використанням інформації Фішера. Отримано вирази для розрахунку дисперсій оцінок параметрів. Наведено рівняння для перевірки існування розв'язку системи рівнянь правдоподібності. Виникнення макротріщин може мати різну фізичну природу, наприклад, накопичення втоми або миттєве пошкодження внаслідок перевантаження, що перевищує розрахункове. Різна фізична природа передбачає різні види профілактичної роботи з їх запобігання та описується різними імовірнісними моделями. Запропоновано критерій відношення правдоподібності для перевірки статистичної гіпотези щодо ситуації раптової появи макротріщин при навантаженні, що перевищує розрахункове.

Отримані результати показують особливості структури досліджуваних вибірок та можливі методи статистичної обробки. Бібліографія 11 назв Таблиця 2.

1. Introduction

The purpose of this work is to study the possibility of statistical processing of one type of grouped samples – interval censored samples with overlapping observation intervals. The study was carried out on the examples of the analysis of damage to the structure of the airframe of an aircraft in operation during inspections. Interval censored samples with overlapping observation intervals describe the durability of the structure before the appearance of an initial crack (macrocrack).

In the absence of interfering events, the observation results will be known for all observations and the sample will be complete (sample with complete data). Censoring is defined as an interfering event that prevents the observation of the event of interest from being obtained, for example, in terms of the size of the initial damage in mm (inches). The observation result can be indicated only in a certain time interval. The boundaries of the observation intervals are determined by the well-established names of individual data: right censored data, left censored data, interval censored data. The difference between these data is only within the boundaries of the observation intervals.

In libraries of data processing programs, for example, Reliability Python library (2021), it is possible to process samples consisting of complete data and supplemented with censored data. In this case, a certain procedure is used for ranking the inspection times in the order of non-decreasing values and grouping the data with non-overlapping intervals. These data are actually interval censored samples with non-overlapping intervals. The methods for processing such samples are standardized. For example, the International Electrotechnical Commission (2008) developed the IEC 61649: 2008 standard, Weibull analysis, for estimation using the Weibull probability distribution model.

The named programs and standards do not cover the statistical processing of interval-censored samples with overlapping observation intervals. This type of sampling was considered in due time (Artamonovskii and Kordonskii, 1970) in relation to the analysis of reliability based on the results of one-time inspections of structures. In the article, the samples are named naturally grouped. Let us indicate the main features of such samples with generalization for one-time and multiple examinations to describe the initial cracks in the structure:

1) the result of the observation can only be described by the time interval on all inspected structures;

2) the sample consists entirely of such interval observations;

3) the number of observations in the sample is equal to the number of observation intervals;

4) each observation in the sample is censored by time intervals of different length;

5) numeric values of interval boundaries can include infinity or zero;

6) the boundaries of the intervals intersect and may overlap arbitrarily.

This type of samples is defined by the object of research in this article. It is interesting for the possibility of analyzing the initial signs of degradation of structures, for example, the appearance of initial cracks (macrocracks). There are certain stages of degradation – the appearance of an initial crack of a small size, then the development of a visible crack up to a size dangerous for safe operation. This time of the appearance of the initial cracks is interesting from the point of view of the development of preventive measures. The appearance of an initial crack can have two different causes. This is the result of a gradual change in the internal state of the structure zone (normal situation for fatigue damage) or as a result of a 'peak load' that exceeds the design. In the second case, preventive maintenance does not improve the design and reinforcement of the structure is necessary.

As a result of the study, a certain understanding of the structure of interval censored samples with overlapping observation intervals was obtained. The basic relations of the maximum likelihood method for estimating the parameters of the probability distribution law and their variances were obtained. An accumulation method is proposed for the computational procedure. A statistical criterion for the likelihood ratio is proposed to assess the cause of the appearance of initial cracks.

2. Methodology & methods

2.1. Methodology

The research methodology was based on determining the possibility of statistical processing of interval-censored samples with overlapping observation intervals and the development of statistical analysis methods.

As an example of such samples, we took the data that are encountered in the analysis of damage to the structure of the airframe of an aircraft in operation. A model of damage development has been constructed to answer the question about the nature of the appearance of initial cracks. Examples of the formation of interval censored samples with overlapping observation intervals in relation to the appearance of initial cracks are considered.

Further, an algorithm for statistical data analysis was developed using the maximum likelihood method for estimating distribution parameters. The method is usually used for complex sample structures. A parametric approach is used for a family of distributions with location and scale parameters: Weibull, lognormal, and normal. The accumulation method is used as a special case of the Newton-Raphson iterative method. The verification of the existence condition for the solution of the likelihood equations is considered. The properties of the maximum likelihood estimates are analyzed. Relationships are obtained for calculating the variances of the maximum likelihood estimates. A likelihood ratio criterion is proposed to test the statistical hypothesis about the possible quasi-static nature of the appearance of macrocracks. The selected methods of statistical processing of interval-censored samples with overlapping observation intervals are suitable for the purpose of the study.

2.2. Damage development model

This section describes the model for the development of damage. The stage of the appearance of an initial crack (macrocrack) is highlighted. The time of occurrence of a macrocrack is a random variable. The size of a macrocrack can practically not be measured either during tests or during inspections

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in service. It can be estimated for the airframe structure of an aircraft in the range from 1mm to the minimum size of the detected cracks. At the stage of the appearance of a macrocrack, it is important to establish the nature of the appearance of damage – material fatigue under the action of an alternating load or a load exceeding the calculated one.

For all complex technical products, there are programs for monitoring the state of the structure in operation. This is the control of aging, strength and corrosion. The rules for examining the zones of the structure, inspection technologies and the dimensions of permissible and maximum damage are determined. As a result, statistical information about damages is accumulated and processed.

We will distinguish between the concepts of damage and failure. Failure is an event consisting in a malfunction of the product. The failure time is recorded accurately. Damage is an event consisting in a violation of the healthy state of an object while maintaining a healthy state and having a certain size, for example, the length of a crack in mm (inches). Damage can be revealed after some time during routine inspections. The damage undergoes irreversible changes and can develop into failure.

Classification of damages, causes of their occurrence and probabilistic models of damageability in the work of Gertsbakh, I. and Kordonskiy Kh. (1969) allows describing the fracture propagation model as follows:

1. Unobservable period – accumulation of damages in the stress concentration zone. Damage at this stage is not detected by non-destructive testing methods (unless only acoustic emission).

2. Macrocrack, initial crack. At this stage, the incubation period of damage accumulation ends, a material discontinuity forms, and crack growth begins. The area of the structure where an initial crack is likely to occur may or may not be known.

The time of appearance is a random variable. The size of a macrocrack practically cannot be measured due to the limited possibilities of non-destructive testing and visual inspection methods, as well as the conditions of testing in operation. The probability of detecting a macrocrack is practically zero. The initial crack size can be estimated as less than the minimum detectable crack size when inspected in service.

In structural design and safety life calculations, e.g. Wood (1971), and also in the work of Tavares, S. M. O. and de Castro, P. M. S. T. (2019) an initial crack size of 0.040-0.070 inches (1-2 mm) is estimated for the airframe frame of an airplane, and this value is less than the minimum detectable crack in the sample.

3. Visible crack, trunk. Detected by non-destructive testing methods and visual inspection. The growth of a visible crack is determined by loads and damage accumulation at the crack tip. Its growth rate depends on the stress state of the place where the crack tip is located. Measurement of the crack length is determined by technological documents. The detected cracks, as a rule, do not exceed the maximum permissible dimensions for safe operation, however, they require analysis in order to prevent them.

4. The nature of the appearance of macrocracks. For fatigue damage, there is a gradual change in the internal state that develops into a macrocrack. It is also necessary to take into account the possibility of the appearance of 'quasi-static damage' – when the appearance of a macrocrack is caused by a sudden single external action that initiated its appearance. This is a situation of random failures, the failure rate is constant and reinforcement of the structure is required taking into account the loads exceeding the design ones. It is necessary to distinguish between these situations, because they have a different physical nature and are described by different statistical models.

2.3. Observations of the current state of products during inspections

This section discusses the schemes of one-time and multiple inspections of structures in operation. Forms of fixing the current state are presented based on the results of visual inspections or non-destructive testing, in which the operating time and the size of the detected crack are indicated. To describe the moment of the appearance of the initial crack, an event indicator is used, which takes values of 0 or 1, depending on the fact of a crack detection in the corresponding observation interval. Based on the inspection results, interval censored samples are formed with overlapping observation intervals for the moments of the appearance of macrocracks.

A one-time inspection of products is defined as a technical inspection of a set of similar products, carried out on special instructions or a bulletin in a specific area of the structure. Table 1 shows typical results from one-time structural inspections. As an operating time, i.e. the lifetime of the product, the operating time from the beginning of operation in flight hours, number of flights or years (months) is taken. The current state is shown in the left half of the table.

The right half of the table describes the events that characterize the observation results – the moments of the appearance of macrocracks τ. The moments of occurrence of macrocracks are censored by intervals, including zero at the event $\{\tau_i < x_i\}$ or infinity on event $\{\tau_i \ge x_i\}$. Observation intervals and event indicators form a sample of interval censored data with overlapping observation intervals.

The second scheme of examinations is typical results of multiple examinations, see Table 2. Inspections are carried out during structural repairs and other forms of maintenance. The operating time of the products is recorded from the start of operation

Table 1

Results of one-time examinations and interval censoring operating time before the appearance of a macrocrack

No	Current state during inspections		Moments of occurrence of a macrocrack τ		
	Inspection time,	Crack lenght, mm	Interval event indicator 0 – no, 1 – yes		Censoring an event
	x_i	δ_i	$\{\tau_i < x_i\}$	$\{\tau_i \geq x_i\}$	τ_i
1.	5181	0	0	1	right c.
2.	4676	0	0	1	right c.
3.	4986	15	1	0	left c.
4.	4998	5	1	0	left c.
5.	4800	0	0	1	right c.
6.	4998	0	0	1	right c.
7.	5002	20	1	0	left c.
8.	6411	0	0	1	right c.

Table 2

Results of multiple examinations and interval censoring of operating time before the appearance of a macrocrack

	Current state during inspections			Moments of occurrence of a macrocrack τ			
	Inspection time, $x_{1i} \leq x_{2i}$		Crack lenght, mm δ_i	Interval event indicator 0 – no, 1 – yes			Censoring an event
	x_{1i}	x_{2i}		$\{\tau_i < x_{1i}\}$	$\{x_{1i} \leq \tau_i < x_{2i}\}$	$\{\tau_i \geq x_{2i}\}$	τ_i
1.	6036	9531	65	0	1	0	interval c.
2.	5181	8181	0	0	0	1	right c.
3.	4678	7296	65.5	1	0	0	left c.
4.	4986	8441	8	0	1	0	interval c.
5.	4998	8068	40	1	0	0	left c.
6.	4800	7668	0	0	0	1	right c.
7.	4998	8086	0	0	0	1	right c.
8.	5002	8319	8.5	0	1	0	interval c.
9.	5002	7900	0	0	0	1	right c.
10.	5004	8366	0	0	0	1	right c.
11.	5100	8265	0	0	0	1	right c.

for the last repair (inspection) x_2 and for the previous repair (inspection) x_1. The size of the detected crack at the time of the next repair is also recorded.

The right half of the table describes the events that characterize the observation results – the moments of the appearance of macrocracks. Inspection times censor the moments of occurrence of macrocracks with three types of intervals described by the events $\{\tau_i < x_{1i}\}$, $\{\tau_i \geq x_{2i}\}$ and $\{x_{1i} \leq \tau_i < x_{2i}\}$. Observation intervals and event indicators form a sample of interval censored data with overlapping observation intervals.

2.4. Maximum likelihood method for distributions with location and scale parameters

This section presents the basic relationships for statistical processing of samples with interval-censored data with overlapping intervals. Maximum likelihood equations are obtained for estimating the parameters of probability distributions with location and scale parameters. An iterative procedure for solving systems of equations by the accumulation method (a special case of the Newton-Raphson iterative method) using the Fisher information matrix is proposed. Expressions for calculating the variances of parameter estimates are obtained. An equation for checking the existence of a solution to the system of likelihood equations is presented. The asymptotic properties of the estimates are considered.

Let us introduce the following designations:

1. x – operating time from the beginning of product operation (product life time), calculated in flight hours, number of flights, months, operating time is considered as a variable.

2. $\tau_1, \tau_2, ..., \tau_N$ – operating time before the appearance of macrocracks. Considered as a

repeated sampling of random variables $T_1, T_2, ..., T_N$, which are independent and equally distributed with an unknown cumulative distribution function $F_\tau(x) \in \mathcal{F}$. The class $\mathcal{F}$ is finite-dimensional and has the form $\mathcal{F} = \left\{ F_x(x; \vec{\theta}),\ \hat{\theta} \in \Theta \subset R^m \right\}$. The cumulative distribution function $F_\tau(x; \vec{\theta})$ is known, but the parameters $\vec{\theta}$ unknown, i.e. parameters are determined. Distribution density function $\frac{\partial F_\tau(x)}{dx} = f_\tau(x),\ \exists f_\tau(x) \geq 0$.

3. $x_{1i}, x_{2i};\ x_{1i} \leq x_{2i};\ i = 1, ..., N$ – inspection moments. In the general case, they are considered as values of random variables $X_{11}, X_{12}, ..., X_{1N};\ X_{21}, X_{22}, ..., X_{2N}$ (for one-time examinations $x_{1i} = x_{2i};\ X_{1i} = X_{2i};\ n_{2i} = 0$). They are independent and equally distributed with an unknown cumulative distribution function $F_x(x) \in \mathcal{F}$, $\mathcal{F} = \left\{ F_x(x; \vec{\theta}),\ \hat{\theta} \in \Theta \subset R^2 \right\}$. Distribution density function $\frac{\partial F_x(x)}{dx} = f_x(x),\ \exists f_x(x) \geq 0$.

4. From physical considerations, we assume that the censored moments of the appearance of macrocracks and the censoring moments of inspections are random and independent. Accordingly, independent random variables $T_1, T_2, ..., T_N$ и с.в. $X_{11}, X_{12}, ..., X_{1N}, X_{21}, X_{22}, ..., X_{2N}$. Then the inspection moments are censoring points for random values $T_1, T_2, ..., T_N$ and are deterministic.

5. n_{ri} indicator (0, 1), characterizing the appearance of a macrocrack in the corresponding interval of inspection times:

$$n_{ri} = \begin{cases} (1,0,0), \ if & \{\tau_i \leq x_{1i}\} \\ (0,1,0), \ if & \{x_{1i} < \tau_i \leq x_{2i}\}, \ i = 1, ..., N;\ r = 1,2,3;\ \sum_r \sum_i n_{ri} = N. \\ (0,0,1), \ if & \{\tau_i > x_{2i}\} \end{cases}$$

The cumulative distribution function for τ is defined through the location and scale parameters θ_1, θ_2 and, by definition, describes the corresponding probabilities of occurrence of the events of interest to us:

$$F_\tau(x, \theta_1, \theta_2) = F_0\left(\frac{\varphi(x) - \theta_1}{\theta_2} \right). \quad (1)$$

The likelihood function is determined at a fixed value of the sample $\tau = (\tau_1, \tau_2, ..., \tau_N)$ and independence of T and X:

$$L = L_\tau(\theta_1, \theta_2) = \prod_{i=1}^{N} F_{1i}^{n_{1i}} \cdot \left(F_{2i} - F_{1i}\right)^{n_{2i}} \cdot \left(1 - F_{2i}\right)^{n_{3i}}. \quad (2)$$

where

$$F_{qi} = P\left\{\tau_i \leq x_{qi}\right\},\ q = 1,2;\quad n_{ri} = \left(n_{1i}, n_{2i}, n_{3i}\right),\ r = 1,2,3,\ i = 1, ..., N.$$

The estimates $\widehat{\theta_1}, \widehat{\theta_2}$ of the parameters θ_1, θ_2 are found by solving the system of maximum likelihood equations (for the scheme of one-time examinations $T_{1i} = T_{2i}$, the second terms are identically equal to zero):

$$\begin{cases} S_1(\vec{\theta}) = 0 \\ S_2(\vec{\theta}) = 0 \end{cases}, \quad (3)$$

where

$$\begin{cases} S_1(\vec{\theta}) = \dfrac{\partial \ln L}{\partial \theta_1} = -\dfrac{1}{\theta_2} \sum_{i=1}^{N} n_{1i} \dfrac{F'_{1i}}{F_{1i}} + n_{2i} \dfrac{F'_{2i} - F'_{1i}}{F_{2i} - F_{1i}} - n_{3i} \dfrac{F'_{2i}}{1 - F_{2i}} \\ S_2(\vec{\theta}) = \dfrac{\partial \ln L}{\partial \theta_2} = -\dfrac{1}{\theta_2} \sum_{i=1}^{N} n_{1i} \dfrac{u_{1i} F'_{1i}}{F_{1i}} + n_{2i} \dfrac{u_{2i} F'_{2i} - u_{1i} F'_{1i}}{F_{1i}} - n_{3i} \dfrac{u_{2i} F'_{2i}}{1 - F_{2i}} \end{cases},$$

$$u_{qi} = \frac{\ln x_{qi} - \theta_1}{\theta_2};\ F'_{qi} = \frac{\partial \ln F_{qi}}{\partial u};\ q = 1,2.$$

Solving the system of maximum likelihood equations provides the maximum likelihood function. We use the computational accumulation method as an iterative procedure for finding a solution. Let us expand the maximum likelihood equations in the vicinity of $\widehat{\theta_1}, \widehat{\theta_2}$ in a Taylor series up to linear terms (for the k – iteration):

$$\begin{cases} \left(\dfrac{\partial \ln L}{\partial \theta_1}\right)_{\hat{\theta}_i} = \left(\dfrac{\partial \ln L}{\partial \theta_1}\right)_{\hat{\theta}_{i(k)}} + \left(\hat{\theta}_1 - \hat{\theta}_{1(k)}\right)\left(\dfrac{\partial^2 \ln L}{\partial \theta_1^2}\right)_{\hat{\theta}_{i(k)}} + \left(\hat{\theta}_2 - \hat{\theta}_{2(k)}\right)\left(\dfrac{\partial^2 \ln L}{\partial \theta_1 \partial \theta_2}\right)_{\hat{\theta}_{i(k)}} = 0 \\ \left(\dfrac{\partial \ln L}{\partial \theta_2}\right)_{\widehat{\theta_i}} = \left(\dfrac{\partial \ln L}{\partial \theta_2}\right)_{\hat{\theta}_{i(k)}} + \left(\hat{\theta}_1 - \hat{\theta}_{1(k)}\right)\left(\dfrac{\partial^2 \ln L}{\partial \theta_2 \partial \theta_1}\right)_{\hat{\theta}_{i(k)}} + \left(\hat{\theta}_2 - \hat{\theta}_{2(k)}\right)\left(\dfrac{\partial^2 \ln L}{\partial \theta_2^2}\right)_{\hat{\theta}_{i(k)}} = 0 \end{cases}. \quad (4)$$

Artamonovskii and Kordonskii (1970) show the asymptotic consistency of the maximum likelihood estimates and the convergence in probability of the second derivatives of the likelihood function to their mathematical expectations. We determine the elements of the Fisher information matrix $I\left(\vec{\theta}\right)$:

$$\begin{cases} I_{11} = \dfrac{1}{\theta_2^2} \sum_{i=1}^{N} n_{1i} \dfrac{(F'_{1i})^2}{F_{1i}} + n_{2i} \dfrac{(F'_{2i} - F'_{1i})^2}{F_{2i} - F_{1i}} + n_{3i} \dfrac{(F'_{2i})^2}{1 - F_{2i}} \\ I_{12} = I_{21} = \dfrac{1}{\theta_2^2} \sum_{i=1}^{N} n_{1i} \dfrac{u_{1i}(F'_{1i})^2}{F_{1i}} + n_{2i} \dfrac{(F'_{2i} - F'_{1i})(u_{2i} F'_{2i} - u_{1i} F'_{1i})}{F_{2i} - F'_{1i}} + n_{3i} \dfrac{u_{2i}(F'_{2i})^2}{1 - F_{2i}} \\ I_{22} = \dfrac{1}{\theta_2^2} \sum_{i=1}^{N} n_{1i} \dfrac{(u_{1i} F'_{1i})^2}{F_{1i}} + n_{2i} \dfrac{(u_{2i} F'_{2i} - u_{1i} F'_{1i})^2}{F_{2i} - F'_{1i}} + n_{3i} \dfrac{(u_{2i} F'_{2i})^2}{1 - F_{2i}} \end{cases} \quad (5)$$

We also define the covariance matrix (inverse information matrix) $I^{-1}\left(\vec{\theta}\right)$:

$$I^{-1}(\vec{\theta}) = \frac{1}{\det[I(\vec{\theta})]} \begin{pmatrix} I_{22} & -I_{12} \\ -I_{21} & I_{11} \end{pmatrix},$$ where $\det[I(\vec{\theta})]$ – determinant of a matrix.

The iterative procedure for finding the maximum likelihood estimates is a function of the accumulations $S_1\left(\vec{\theta}_{(k)}\right), S_2\left(\vec{\theta}_{(k)}\right)$ and the Fisher information matrix, where the unknown parameters are replaced by $\theta_{1(l+1)}, \theta_{2(k+1)}$:

$$\theta_{i(k+1)} = \theta_{i(k)} + \delta\theta_{i(k+1)},\ i = 1,2;\ k = 0,,1,...,$$

$$\delta\theta_{1(k+1)} = \frac{I_{12}\left(\vec{\theta}_{(k)}\right) \cdot S_2\left(\vec{\theta}_{(k)}\right) + I_{22}\left(\vec{\theta}_{(k)}\right) \cdot S_1\left(\vec{\theta}_{(k)}\right)}{\det[I(\vec{\theta}_{(k)})]}, \quad (6)$$

$$\delta\theta_{2(k+1)} = \frac{I_{12}\left(\vec{\theta}_{(k)}\right) \cdot S_1\left(\vec{\theta}_{(k)}\right) + I_{11}\left(\vec{\theta}_{(k)}\right) \cdot S_2\left(\vec{\theta}_{(k)}\right)}{\det[I(\vec{\theta}_{(k)})]}.$$

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An important point is the possibility of using the check for the existence of a solution to the system of maximum likelihood equations. Artamonovskii and Kordonskii (1970) proposed a criterion for checking the samples under study:

$$\frac{\sum_{i=1}^{N}(n_{1i}+n_{2i})}{\sum_{i=1}^{N}n_{3i}} > \frac{\sum_{i=1}^{N}(n_{1i}\cdot \ln x_{1i})+\sum_{i=1}^{N}(n_{2i}\cdot \ln x_{2i})}{\sum_{i=1}^{N}(n_{3i}\cdot \ln x_{2i})}. \quad (7)$$

The results of the research (Makarov, Martinov and Rastrigin, 1981) showed the effectiveness of the estimation procedure using the accumulation method. The number of iterations was compared with respect to the procedure for applying the method of grids. The choice of the initial approximations $\theta_{1(0)},\theta_{2(0)}$ takes into account the peculiarities of the structure of the initial observation data during examinations. It takes up to 10 iterations to achieve the specified precision, $\{|S_1|,|S_2|\}\le \varepsilon,\ \varepsilon = 1\cdot 10^{-6}$.

In the studies of Rastrigin (1983, 2000), the properties of maximum likelihood estimates were established for observation schemes of states during single and multiple examinations:

1) asymptotic consistency of the estimate

$$\hat{\theta}_i \xrightarrow[N\to\infty]{p} \theta_i,\ i=1,2,$$

2) asymptotic efficiency at which the variances of the parameter estimates correspond to the lower bound of the generalized Rao-Cramer inequality

$$D\{\hat{\theta}_1\}=\frac{I_{22}(\vec{\theta})}{\det[I(\vec{\theta})]};\ D\{\hat{\theta}_2\}=\frac{I_{11}(\vec{\theta})}{\det[I(\vec{\theta})]}, \quad (8)$$

3) asymptotic normality of estimates

$$\vec{\theta} \sim N(\vec{\theta}, I^{-1}[\vec{\theta}]).$$

5.1. Testing the hypothesis of quasi-static damage.

In this section, we solve the problem of testing the hypothesis that the sample belongs to the exponential law of operating time before the appearance of macrocracks τ with the alternative of belonging to the Weibull distribution. When analyzing the current state of products, it is important to check the situation of the appearance of macrocracks due to a sudden external influence. In this case, further statistical analysis is not desirable, the structure does not withstand peak loads and the structure needs reinforcement. Predictive maintenance in such a damaged area does not improve design. A likelihood ratio criterion is proposed.

To solve the problem, we express the initial Weibull distribution with the parameters β, η in terms of the parameters θ_1,θ_2:

$$F_\tau(\theta_1,\theta_2)=1-\exp\left[-\left(\frac{x}{\eta}\right)^{\beta}\right]= \\ =1-\exp\left[-\exp\left(\frac{\ln x-\theta_1}{\theta_2}\right)\right],\ \theta_1=\ln\eta,\ \theta_2=\frac{1}{\beta}. \quad (9)$$

where β is the parameter of the distribution shape, shows the rate of change in the failure rate; η – resource, operating time, during which 63.2% of macrocracks will appear.

The hypothesis $H_0:\beta=1$ is tested with the alternative $H_1:\beta\ne 1$. Earlier (Kendall and Stuart, 1961) used the likelihood ratio test to test hypotheses. For our problem, we define the statistics S for the likelihood ratio criterion

$$S=-2\ln\frac{L_\tau(\hat{\hat{\theta}}_1,\theta_2=1)}{L_\tau(\hat{\theta}_1,\hat{\theta}_2)}, \quad (10)$$

where

$L_\tau(\hat{\hat{\theta}}_1,\theta_2=1)$ is the likelihood function of the Weibull distribution, $\theta_2=1$, $\hat{\hat{\theta}}_1$ is the estimate of the parameter $\hat{\theta}_1$, which converts the likelihood function to the maximum;

$L_\tau(\hat{\theta}_1,\hat{\theta}_2)$ is the likelihood function of the Weibull distribution with estimates $\hat{\theta}_1,\hat{\theta}_2$,

$L_\tau(\hat{\hat{\theta}}_1,\theta_2=1)$ – actually represents the likelihood function $L_\tau(\lambda)$ of the exponential distribution:

$$F_\tau(\lambda)=1-\exp\left[-\exp\left(\frac{\ln x-\theta_1}{\theta_2}\right)\right]=1-e^{-\lambda x},\ \lambda=1/\theta_1\ \ \theta_2=1 \quad (11)$$

Find a new estimate $\hat{\hat{\theta}}_1$ of the parameter $\hat{\theta}_1$ for $\hat{\theta}_2=1$ from the equation:

$$S_1=\frac{\partial L_\tau(\hat{\hat{\theta}}_1,\theta_2=1)}{\partial\theta_1}=-\sum_{i=1}^{N}n_{1i}\frac{F'_{1i}}{F_{1i}}+n_{2i}\frac{F'_{2i}-F'_{1i}}{F_{2i}-F_{1i}}-n_{3i}\frac{F'_{2i}}{1-F_{2i}}=0, \quad (12)$$

where

$$u_{qi}=\frac{\ln x_{qi}-\theta_1}{\theta_2};\ F'_{qi}=\frac{\partial\ln F_{qi}}{\partial u};\ q=1,2.,$$

$$F_{qi}=P\{\tau_i\le x_{qi}\}=F_\tau(\hat{\theta}_1,\theta_2=1),\ q=1,2;$$

$$i=1,\dots,N;\ n_{ri}=(n_{1i},n_{2i},n_{3i}),\ r=1,2,3.$$

The statistics S in (10) under the hypothesis H_0 is asymptotically distributed over $\chi^2_{1;0.05}$ with 1 degree of freedom. Under the hypothesis, one parameter β is checked, then $r=1$; distribution has $r+s=2$ parameters, $s=1$, because $s\ne 0$, then the hypothesis is composite. Critical region at a significance level of 0.05 $S>\chi^2_{1;0.05}=3.841$.

The estimates of the parameters β, η and their variances are determined in terms of the parameters θ_1, θ_2:

$$\hat{\beta} = \frac{1}{\hat{\theta}_2};\ D\{\hat{\beta}\} = \frac{1}{\hat{\theta}_2^4} D\{\hat{\theta}_2\};\ \hat{\eta} = e^{\hat{\theta}_1};\ D\{\hat{\eta}\} = \exp\{2\hat{\theta}_1\} \cdot D\{\hat{\theta}_1\}. \quad (13)$$

Example. For illustration, we present the results of data processing in Table 2 of inspections of the current state of structures to observe the moments of the appearance of macrocracks. Checking the conditions for the existence of a solution (7) of the system of equations (3) gives the result 0.833>0.822, i.e. is a solution to the system of likelihood equations. The solution of the system of equations (3) for the Weibull distribution (9) using equations (6), (13) gives the following results:

$$\hat{\theta}_1 = 9.237,\ \hat{\theta}_2 = 0.429,\ Var\left(\widehat{\theta_1}\right) = 0.159,\ Var\left(\widehat{\theta_2}\right) = 0.187,$$
$$\hat{\eta} = 10270.,\ \hat{\beta} = 2.331,\ Var\left(\hat{\eta}\right) = 1.6804 + E7,\ Var\left(\hat{\beta}\right) = 5.317.$$

Testing the hypothesis about the quasi-static nature of the appearance of macrocracks $H_0 : \beta = 1$ with the alternative $H_0 : \beta \neq 1$ according to the likelihood ratio criterion (10) gives the following result:

$$S = -2\ln \frac{L_\tau\left(\hat{\hat{\theta}}_1, \theta_2 = 1\right)}{L_\tau\left(\hat{\theta}_1, \hat{\theta}_2\right)} = -2\left(-11.646 - \left(-10.717\right)\right) = 1.858\,.$$

Then $S < \chi^2_{1;0,05}$, hence the hypothesis H_0 is not rejected. This may be due to the low accuracy of estimating the parameter β in this sample.

6. Results and Discussion

Results. The result of the study is the methods of statistical processing of specific data that are accumulated during inspections in operation. The object of research is determined – the type of grouped samples, interval censored samples with overlapping observation intervals. Examples of samples entirely consisting of interval data are given. A model of damage development has been developed.

An algorithm and statistical methods for processing the samples under study have been developed. The method of maximum likelihood is applied to find the parameters of the distribution of the time of occurrence of the initial cracks and their dispersions. Basic equations of maximum likelihood are obtained for probability distributions with location and scale parameters. The solution of systems of equations is performed by the accumulation method (a special case of the Newton-Raphson iterative method) using the Fisher information matrix. Expressions for calculating the variances of parameter estimates are obtained. An equation for checking the existence of a solution to the system of likelihood equations is presented. The maximum likelihood estimates for the studied samples are asymptotically consistent, efficient, and normally distributed. A likelihood ratio criterion is proposed for checking the studied samples for belonging to an exponential distribution.

Discussion. The discussed research topic of interval-censored samples with overlapping intervals appeared in the analysis of data from structural inspections to assess the development of damage. The period of the appearance of initial cracks, which cannot be detected by means of non-destructive testing for various reasons, is especially highlighted. This period determines the nature of damage occurrence (fatigue or overload), and also determines the initial conditions when calculating the growth rate of fatigue cracks. The importance of understanding the specifics of the processed samples in the form of fully censored observations of the appearance of initial cracks is emphasized.

The need to develop a mathematical apparatus is due to the lack of software for the samples under study. The reason is the lack of data on the exact moments of the appearance of initial cracks at least on some structures, which increase the information content of the initial data and allow the use of standard tools.

It is possible to analyze the initial signs of structural degradation through statistical processing of samples in the form of interval censored samples with overlapping observation intervals. The mathematical apparatus in the form of the maximum likelihood method is used to estimate the distribution parameters of the time of occurrence of macrocracks. The choice of the distribution law for a specific analysis is established from the physics of the damage process.

Interval censored samples with intersecting intervals characterize the durability of the structure before the appearance of an initial crack (macrocrack). Further, the main indicators of the reliability and durability of the structure can be calculated. The research results can be used to analyze the appearance of initial damage during wear and corrosion, as well as to analyze the survival rate of various research objects, taking into account the initial changes in the situation.

7. Conclusions

The conclusions of the study are presented. In general, the goal of the study was achieved.

The structure of interval-censored samples with overlapping observation intervals is determined. Their features in relation to other types of grouped samples are described. Examples of the formation of the studied samples according to the primary information of the results of inspections of structures in operation are considered. A damage development model has been built to answer the question about the appearance of initial cracks that cannot be detected due to the limited possibilities of non-destructive testing methods and visual inspection, as well as inspection conditions in operation. The appearance of initial cracks leads to the formation of statistical samples in the form of interval-censored samples with intersecting intervals.

A parametric approach is proposed for the statistical analysis of the studied samples. The maximum likelihood method was used to find estimates for the parameters of the probabilistic law of distribution of continuous events, for example, the time of appearance of initial cracks (macrocracks) in the structure. To solve the system of likelihood equations, the accumulation method was applied to find estimates of the parameters and their variances. To establish the nature of the appearance of initial cracks, a statistical criterion of the likelihood ratio is proposed.

The presented mathematical apparatus can be useful for other areas of research, when the structure of the initial data coincides with interval censored samples with overlapping observation intervals. It is also possible that applying a parametric approach to a solution is not the only solution.

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